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ON INVARIANT CONFIDENCE INTERVALS FOR THE PARAMETERS OF A NEW LIFE DISTRIBUTION
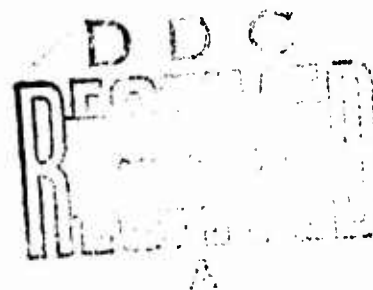
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A statistic which is asymptotically independent of the shape parameter α is shown to be capable of providing confidence bounds for the scale parameter β. These bounds, however, are subsequently shown to exhibit invariance with respect to α only for samples of size fifty or larger.

Finally, three statistics based on maximal invariants for tests and confidence sets independent of α are investigated in terms of accuracy of confidence bounds for (power of tests concerning) β. Percentage points of the statistic yielding the most accurate confidence bounds for β, among those investigated, are tabulated for samples of size n, n = 2(1)5.

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Foreword

The research contained in this report which was performed at Rocketdyne was sponsored by the Aerospace Research Laboratories, Air Force Systems Command, U.S. Air Force, Wright-Patterson Air Force Base, Ohio, under Contract No. F33615-70-C-1216 and was a part of Project 7071, Research in Applied Mathematics, technically monitored by Dr. H. Leon Harter.

Abstract

In this report we explore the problem of obtaining confidence intervals and tests of hypotheses for the parameters of a new family of life distributions derived in [1]. Several statistics whose distributions are independent of the shape parameter are investigated in terms of accuracy of confidence bounds for the distribution scale parameter. Percentage points of the statistic yielding the most accurate confidence bounds on the scale parameter, among those investigated, are tabulated for samples of size n, $n=2(1)5$.

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0. Introduction

In [1], Birnbaum and Saunders derived a new family of life distributions from plausible considerations of the physical behavior of fatigue crack growth under repeated loading. Certain point estimators for the parameters of this family of distributions were investigated in [2]. In the following, we explore the problem of obtaining confidence intervals and tests of hypotheses for these parameters.

We denote by $\mathcal{G}(\alpha, \beta)$ the two-parameter distribution of a non-negative random variable derived in [1] and defined by

$$(0.1) \qquad \mathfrak{N}[\frac{1}{\alpha} \xi(t/\beta)] \qquad \text{for } t > 0$$

where $\alpha > 0$, $\beta > 0$ and

$$(0.2) \qquad \xi(t) = \sqrt{t} - \frac{1}{\sqrt{t}} ,$$

and $\mathfrak{N}$ is the distribution function of the standard normal variate. The parameter α determines the shape of the distribution while the median β is a scale parameter. Iterative numerical procedures for the computation of the maximum likelihood estimates of α and β are developed and a simple estimator $\tilde{\beta}$ of β is derived in [2]. We show here that the maximum likelihood estimators, and the estimator $\tilde{\beta}$ as well, even though they have optimal asymptotic properties, are such that they cannot form the basis for invariant tests and confidence intervals for either of the parameters for all sample sizes. Several statistics whose distributions are independent of the shape parameter α are investigated in terms of their consequent confidence regions

for the scale parameter σ. The percentage points of the most acceptable statistics among those investigated are evaluated by Monte Carlo methods and the results presented. Further, some comparisons of power are also obtained by the same means and the curves exhibited.

1. Maximum Likelihood Estimators

In order to define the maximum likelihood estimators of α and β we first define for a given set of positive random variables $T_1, \ldots, T_n$ the arithmetic and harmonic means, respectively

$$S = \frac{1}{n} \sum_{i=1}^{n} T_i, \qquad R = \left[\frac{1}{n} \sum_{i=1}^{n} \frac{1}{T_i} \right]^{-1}$$

and the harmonic mean function

$$K(x) = \left[\frac{1}{n} \sum_{i=1}^{n} (x+T_i)^{-1} \right]^{-1}.$$

Then if $T_1, \ldots, T_n$ represents a sample of independent random variables each distributed as $\mathcal{G}(\alpha, \beta)$, the maximum likelihood estimator $\hat{\beta}$ of β is the unique positive solution of the random equation $g(x) = 0$ where

$$g(x) = x^2 - x[2R + K(x)] + R[S + K(x)].$$

The maximum likelihood estimator $\hat{\alpha}$ of α is then given in terms of $\hat{\beta}$, S, R and

$$\hat{\alpha} = \left[\frac{1}{n} \sum_{i=1}^{n} \xi^2 \left(\frac{T_i}{\hat{\beta}} \right) \right]^2 = [\frac{S}{\hat{\beta}} + \frac{\hat{\beta}}{R} - 2]^{\frac{1}{2}}.$$

These results were given in [2].

The simplified estimator $\tilde{\beta}$ of β, which is shown in [2] to be consistent and for small values of α virtually the same as the maximum likelihood estimator $\hat{\beta}$ is given by

$$\hat{\beta} = \sqrt{SR}.$$

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If the sample size is $n = 2$, then it also follows from the results of [2] that the maximum likelihood estimators of α and β satisfy

$$\hat{\alpha} = \left| \xi \left(\sqrt{\frac{T_1}{T_2}} \right) \right| , \qquad \hat{\beta} = \tilde{\beta} = \sqrt{T_1 T_2} \ .$$

We note that for $n = 2$, the maximal invariant for confidence sets and tests of hypotheses independent of α is the ratio $\xi(T_2/\cdot)/\xi(T_1/\cdot)$. For $n = 2$ there is apparently no function of $\hat{\beta}/\beta$ and $\hat{\alpha}$ to which this ratio is equal. One would infer that functions of $\hat{\alpha}$ and $\hat{\beta}$ (or $\tilde{\beta}$) cannot be used to obtain confidence sets for β. In fact, we now demonstrate that for $n = 2$ one cannot have confidence sets and tests of hypotheses for either α or β based on functions of $\hat{\alpha}$ and $\hat{\beta}$.

Let X_i denote a random variable with law $\mathcal{G}(\alpha,1)$. In distribution we see that

$$(1.1) \qquad \frac{\hat{\beta}}{\beta} = \sqrt{X_1 X_2} = \frac{1}{\sqrt{X_1 X_2}} = \frac{\beta}{\hat{\beta}} \ ,$$

since each X_i is equal in distribution to its own reciprocal. This property we call the *reciprocal property*. Thus (1.1) says that $\hat{\beta}/\beta$ has the reciprocal property. We now find the distribution of the product $X_1 X_2$, call it G:

$$G(t) = P[X_1 X_2 \leq t] = P[X_1 \leq t X_2]$$

$$= \int_0^\infty \mathfrak{N}[\tfrac{1}{\alpha} \xi(tx)] d \, \mathfrak{N}[\tfrac{1}{\alpha} \xi(x)].$$

Letting $\psi = \xi^{-1}$ we see

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(1.2) $\qquad G(t) = \displaystyle\int_{-\infty}^{\infty} \mathfrak{N}[\frac{1}{\alpha}\,\xi t\psi(\alpha y)]\,d\,\mathfrak{N}(y).$

From (1.1) it follows that

(1.3) $\qquad P[\frac{\hat{\beta}}{\beta} \le y] = P[X_1 X_2 \le y^2] = G(y^2).$

Since in distribution $\hat{\alpha} = |\xi(\sqrt{X_1 X_2})|$ we see

(1.4) $P[\hat{\alpha} \le z] = P[-z \le \xi\sqrt{X_1 X_2} \le z] = P[\psi(-z) \le \sqrt{X_1 X_2} < \psi(z)]$

$\qquad\qquad = G[\psi^2(z)] - G[\psi^2(-z)].$

Thus in terms of the distribution of G, which contains α as a nuisance parameter, we can express via (1.4) and (1.3) the distributions of $\hat{\alpha}$ and $\hat{\beta}$. Note how the parameter α is "scrambled" in the distribution of $\hat{\alpha}$, in fact, it appears to be almost inextricably bound.

Using the fact that

$$\xi(t) = 2\,\sinh(\frac{\ln t}{2}) \qquad \text{for } t > 0,$$

an identity given in [3], and some straightforward, but tedious, algebra one finds

(1.5) $\qquad \dfrac{1}{\alpha}\,\xi[t\psi(\alpha y)] = \dfrac{2}{\alpha}[\sinh\dfrac{t}{2} + \sinh^{-1}(\dfrac{\alpha y}{2})]$

$\qquad\qquad\qquad = y\,\sinh(\dfrac{t}{2}) + \sqrt{(\dfrac{2}{\alpha})^2 + y^2}\,\cosh(\dfrac{t}{2}).$

Thus we conclude that for exact inference the distribution of $\hat{\beta}$, based on two observations, would be of use only if α were known. Since

$$\dfrac{1}{\alpha}\,\xi[t\psi(\alpha y)] \ge y\,\sinh(\dfrac{t}{2}) + |y|\,\cosh(\dfrac{t}{2}) = \ell(t,y)$$

say, we have

$$(1.6) \qquad G(t) \geq \int_{-\infty}^{\infty} \mathfrak{N}[\ell(t,y)] d\mathfrak{N}(y).$$

Thus finding a $t_{1-\epsilon}$ such that the right-hand side of (1.5) equals $1-\epsilon$ would enable one to obtain a conservative lower confidence bound for β (which is the one of primary interest in life studies) since by (1.3)

$$(1.7) \qquad P[(\hat{\beta}/\sqrt{t_{1-\epsilon}}) \leq \beta] = G(t_{1-\epsilon}) \geq 1-\epsilon.$$

This bound is of level $1-\epsilon$ and invariant with respect to α. Whether or not it is of use would depend upon the outcome of numerical tabulation and the practical situation.

The conjecture that for all samples of size greater than two, the maximum likelihood estimators of both $\hat{\alpha}$ and $\hat{\beta}$ would contain the nuisance parameter α would seem to be substantiated. At the same time a result as explicit as (1.2) and (1.6) appear to be a formidable task for large sample sizes.

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2. Invariant Confidence Intervals and Tests for β

Suppose we have a sample $T_1, \ldots, T_n$ of independent $\mathcal{G}(\alpha, \beta)$ variates. From the equations (0.1) and (0.2) it follows that

$$Z_i = \frac{1}{\alpha} \, \xi \left(\frac{T_i}{\beta} \right) \quad \text{for} \quad i = 1, \ldots, n$$

are independent standard normal variates. Maximal invariants for estimators, tests and confidence bounds for β, invariant with respect to α, are functions of ratios of such variates. We can try to utilize well-known results for the normal distribution to obtain bounds on β functionally independent of α.

First let us consider, for $k + m = n$, the statistic

$$U_{k,m} = \left(\frac{1}{k} \sum_1^k Z_i^2 \right) \Bigg/ \frac{1}{m} \sum_{k+1}^n Z_i^2$$

which has Snedecor's F-distribution with k and m degrees of freedom in numerator and denominator, respectively.

From standard tabulations we can find δ_1, δ_2 such that

$$P[\delta_2 < U_{k,m} < \delta_1] = \gamma$$

for any prescribed $\gamma \in (0,1)$.

The question is whether these percentage points of the F-distribution can be used to obtain confidence bounds on the unknown parameter β independent of α. Using the fact that $\xi^2(x) = x + \frac{1}{x} - 2$ we study an observed value $u_{k,m}(\beta)$ as a function of β and obtain

$$u_{k,m}(\beta) = \frac{\frac{1}{k} \sum_{i=1}^k \xi^2(t_i/\beta)}{\frac{1}{m} \sum_{k+1}^n \xi^2(t_i/\beta)} = \frac{s_1 + (\beta^2/r_1) - 2\beta}{s_2 + (\beta^2/r_2) - 2\beta}$$

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where $s_1 = \frac{1}{k} \sum_1^k t_i$, $r_1 = \left(\frac{1}{k} \sum_{i=1}^k t_i^{-1} \right)^{-1}$ are the arithmetic and

harmonic means of the first k observations, with s_2, r_2 defined

similarly. Consider the quadratic equation representing either the

numerator or denominator. One sees that it has roots $r_i \left(1 \pm \sqrt{1 - \frac{s_i}{r_i}} \right)$

but from the well-known inequality between r, s we infer that both

roots are complex. Therefore, no real roots or poles exist for the

rational function $g(\beta)$ for $\beta > 0$. Unfortunately, closer examination

shows that the distribution of $U_{k,m}$ is virtually invariant with

respect to β so that inversion to obtain meaningful confidence bounds

is not possible. To see this note that

$$u_{k,m}(0) = (s_1/s_2), \qquad u_{k,m}(\infty) = (r_2/r_1)$$

which both have an expected value near unity. Now

$$\text{sgn } u'_{k,m}(\beta) = \text{sgn}(a\beta^2 + b\beta + c)$$

where $a = \frac{1}{r_2} - \frac{1}{r_1}$, $b = \frac{s_2}{r_1} - \frac{s_1}{r_2}$, $c = s_1 - s_2$. Thus the function

$u_{k,m}(\beta)$ has one local maximum and one local minimum but the actual

behavior depends upon the values of a, b, c. Note that a will always

be near zero so that the quadratic equation is nearly linear. The roots

are

$$\frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{|b|}{2a} (\text{sgn } b \pm 1) \mp \frac{c}{|b|} \mp \frac{ac^2}{|b|^3} + O(a^2).$$

Examination shows that one of the roots is near $\left| \frac{c}{b} \right|$ for $|a|$ small.

Some actual computer runs on the IBM-360 were made for the case

$k = m = 50$ and yielded the following values:

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sup $u_{k,m}(\beta)$	inf $u_{k,m}(\beta)$
1.1941	.5968
1.8273	.8473
1.3341	.9117

The range of values was very small. This idea was then abandoned.

Consider the t-distribution with n degrees of freedom

$$V_n = \sum_{i=1}^{n} \frac{z_i}{\sqrt{n}\ \alpha} \Bigg/ \sqrt{\sum_{i=1}^{n} \frac{z_i^2}{n\alpha^2}}\ .$$

Again using the definition of the function ξ we examine an observed value as a function of β, namely

$$v_n(\beta) = \frac{\Sigma\xi(t_i/\beta)}{\sqrt{\Sigma\xi^2(t_i/\beta)}} = \frac{\sqrt{np} - \beta\sqrt{\frac{n}{q}}}{(s + \frac{\beta^2}{r} - 2\beta)^{\frac{1}{2}}}$$

where s,p,q,r are the means of $(t_1,\ldots,t_n)$ of order 1, $\frac{1}{2}$, $-\frac{1}{2}$, -1, respectively.

One checks that

$$\text{sgn } v_n'(\beta) = \text{sgn}[\beta(\sqrt{\tfrac{n}{q}} - \tfrac{\sqrt{np}}{r}) + \sqrt{np} - s\sqrt{\tfrac{n}{q}}]$$

and thus v_n' changes sign only once. For certain cases $v_n(\beta)$ is monotone in β but again the total variation is very small. Consequently, we see that invariance with respect to α virtually forces an invariance with respect to β, at least in the cases examined. We now try to allow some dependence of our statistic upon α for finite sample sizes.

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Hence we seek a statistic whose distribution is invariant with respect to β, depends upon α for finite sample sizes but does not depend upon α asymptotically. Let us try

$$\sum_{i=1}^{n} \xi(T_i/\beta) / [\sum_{i=1}^{n} \xi^2(T_i/\tilde{\beta})]^{\frac{1}{2}}$$

where $\tilde{\beta} = \sqrt{SR}$ is the mean mean, i.e. the geometric mean of the harmonic and arithmetic means Our rationale for this choice is that

$$(\hat{\alpha})^2 = \frac{1}{n} \sum_{i=1}^{n} \xi^2(T_i/\hat{\beta})$$

is the maximum likelihood estimator of α^2 if $\hat{\beta}$ is the maximum likelihood estimator of β. For small α we know by previous work that the mean mean is the same as the maximum likelihood estimator but much easier to compute. This we put in the denominator and obtain

$$(\tilde{\alpha})^2 = \frac{1}{n} \sum_{i=1}^{n} \xi^2(T_i/\tilde{\beta}) = \xi^2(\sqrt{\tfrac{S}{R}}).$$

Since

$$E \frac{S}{R} = \frac{1}{n^2} E \sum_{i,j=1}^{n} \frac{T_i}{T_j} = \left(1 + \frac{\alpha^2}{2}\right)^2 + O(\tfrac{1}{n})$$

we see that as $n \to \infty$ with probability one

$$\xi(\sqrt{\tfrac{S}{R}}) \to \xi(1 + \frac{\alpha^2}{2}) = \alpha^2/\sqrt{4 + 2\alpha^2} .$$

Thus the proposed statistic cannot be invariant with respect to α even asymptotically. Consequently, we propose a modified statistic which has

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a distribution independent of β and asymptotically independent of α, namely

$$Y_n = \sqrt{\frac{n-1}{2n}} \; \frac{\displaystyle\sum_{i=1}^{n} \xi(T_i/\beta)}{\sqrt{\dfrac{S_n}{R_n} - 1}} \; .$$

The coefficient has been chosen to make $S/R - 1$ an unbiased estimator of α^2. Consider an observed value as a function of β

$$y_n(\beta) = c_n \; \frac{\sqrt{\dfrac{p}{\beta}} - \sqrt{\dfrac{\beta}{q}}}{\sqrt{\dfrac{s}{r} - 1}}$$

It is clear that as a function of β it is not only monotone decreasing but has a range of ∞ to $-\infty$. Thus it could be inverted to obtain confidence intervals for β if percentage points were tabulated. Unfortunately, simulation by Monte Carlo procedure of the distribution of Y_n revealed that although the statistic is indeed asymptotically independent of α, the independence is not exhibited for samples of size 50 or smaller.

We now consider the ratio, letting $k+m=n$, of $W = \dfrac{Z_2}{Z_1}$ where $Z_1 = \dfrac{1}{\sqrt{m}} \displaystyle\sum_{i=k+1}^{n} \xi(\dfrac{T_i}{\beta})$, $Z_2 = \dfrac{1}{\sqrt{k}} \displaystyle\sum_{i=1}^{k} \xi(\dfrac{T_i}{\beta})$, and W has the distribution of the ratio of two independent standard normal variates. This distribution is the Cauchy distribution F given by

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan x \qquad -\infty < x < \infty.$$

Alternatively, letting $F_{1,1}$ denote Snedecor's F-distribution with one degree of freedom in numerator and denominator, we have

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$$P[-u < W < u] = P[W^2 \leq u^2] = F_{1,1}(u^2)$$

$$= F(u) - F(-u) = 2F(u) - 1.$$

The last equality follows since $-W$ has the same distribution as W. Thus tables of the $F_{1,1}$ distribution could be used as percentage points.

Consider an observed value of W as a function of β:

$$w(\beta) = (\beta\sqrt{\frac{k}{q_2}} - \sqrt{kp_2})/(\beta\sqrt{\frac{m}{q_1}} - \sqrt{mp_1}).$$

We notice that

$$w(0) = \sqrt{\frac{k}{m}\frac{p_2}{p_1}} , \qquad w(\infty) = \sqrt{\frac{k}{m}\frac{q_1}{q_2}}$$

and

$$sgn \ w(\beta) = \sqrt{mk} \ sgn(\sqrt{\frac{p_2}{q_1}} - \sqrt{\frac{p_1}{q_2}}).$$

Thus the function is either everywhere increasing or everywhere decreasing, and we can distinguish exactly two cases:

Case I. $p_1q_1 > p_2q_2$

Figure 1.

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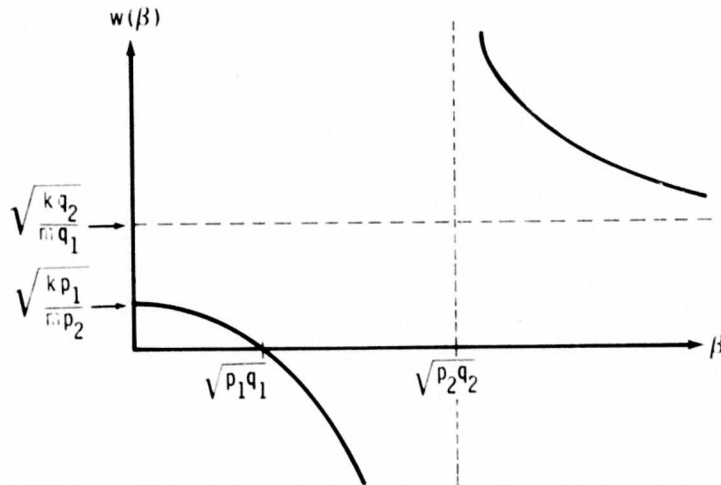
Case II. $p_1 q_1 \cdot p_2 q_2$
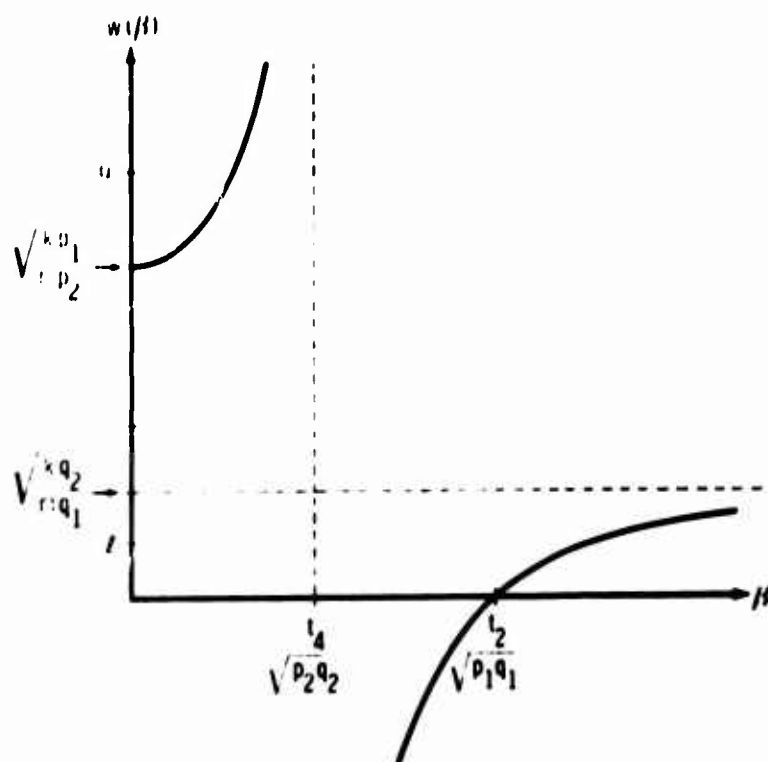
Figure 2.

Suppose it is known that for some ε we have $P[l < W < u] = 1-\varepsilon$; then $\{\beta > 0: l < w(\beta) < u\}$ is a $1-\varepsilon$ level confidence region. By inspection we see that any set of the form $\{\beta > 0: l < w(\beta) < u\}$ could consist of two critical regions, an upper critical region or a lower critical region only, or the empty set which would occur at random depending upon the values of $p_1 q_1$, $p_2 q_2$ in relation to l, u.

This is an unsatisfactory state of affairs since we should like not only to be able to determine one-sided confidence intervals but to be able to place half of the confidence that β lies in the upper critical region and half that it lies in the lower. Moreover, as a matter of practical convenience, we should prefer that one calculation suffice to determine both upper and lower confidence intervals. To this end, we propose to find a method of separating these regions. One realizes that this is equivalent to ordering the observations.

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3. Tests and Confidence Sets for β Based on Ordered Observations

We observe that $T_{i,n}/\beta < 1$ implies

$$X_{i,n} = \xi(T_{i,n}/\beta) = (T_{i,n}/\beta)^{\frac{1}{2}} - (\beta/T_{i,n})^{\frac{1}{2}} < 0$$

and $T_{i,n}/\beta > 1$ implies $X_{i,n} > 0$. We observe also that $\xi(T_{i,n}/\beta)$ is an order-preserving transformation. We note that for a sample of size 2, $\text{Prob}[X_{2,n} < 0] = \text{Prob}[X_{1,n} > 0] = (1/2)^2$. Hence $\text{Prob}[\beta > T_{2,n}] = \text{Prob}[\beta < T_{1,n}] = (1/2)^2$. We can therefore determine iteratively, by Monte Carlo procedures, a value $\underline{u}(\epsilon)$ such that the joint event $X_{2,2}/X_{1,2} > u(\epsilon)$ and $X_{2,2} < 0$ occurs with probability ϵ. Then, an upper confidence bound on β at level $1-\epsilon$ is given by $\max[T_{2,2}, g]$, where g is equal to $[u(\epsilon)T_{1,2} - T_{2,2}]/[u(\epsilon)/T_{1,2} - 1/T_{2,2}]$. Because of the symmetry of the normal distribution, a lower confidence bound on β at level $1-\epsilon$ is given by $\min[T_{1,2}, g'[u(\epsilon), T_{1,2}, T_{2,2}]]$ with $g' = [u(\epsilon)T_{2,2} - T_{1,2}]/[u(\epsilon)/T_{2,2} - 1/T_{1,2}]$. Clearly, because $\text{Prob}[X_{1,n} > 0] = (1/2)^n$, procedures based on the assumption $X_{1,n} > 0$ or $X_{n,n} < 0$ will encounter difficulties if n is large, unless $1-\epsilon$ is extremely close to 1. One might also consider a test based on the joint event $X_{2,2}/X_{1,2} > u(\epsilon)$ and $X_{1,2} > 0$ or the joint event $X_{1,2}/X_{2,2} > u(\epsilon)$, $X_{2,2} < 0$.

For samples of size n one can, therefore, determine $\ell(\epsilon)$ such that the probability that the event

$$nS(\beta) = \sum_{i=1}^{n} X_{i,n} < 0 \quad \text{and} \quad W_1(\beta) = \left(\frac{1}{\sqrt{n-1}}\right) \sum_{i=2}^{n} X_{i,n}/X_{1,n} > \ell(\epsilon)$$

occurs is ϵ. Then an upper confidence bound on β at level $1-\epsilon$ is given by

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$$\eta_n = \max\left[\sum_{i=1}^{n} T_{i,n}^{\frac{1}{2}} / \sum_{j=1}^{n} (1/T_{j,n})^{\frac{1}{2}}, \quad h[\ell(\epsilon), T_{1,n}, \ldots, T_{n,n}]\right], \quad (3.1)$$

where $h[\ell(\epsilon), T_{1,n}, \ldots, T_{n,n}] = [\sqrt{n-1}\, \ell(\epsilon) T_{1,n}^{\frac{1}{2}} - \sum_{i=2}^{n} T_{i,n}^{\frac{1}{2}}]/$
$[\sqrt{n-1}\, \ell(\epsilon)/T_{1,n}^{\frac{1}{2}} - \sum_{i=2}^{n} (1/T_{i,n})^{\frac{1}{2}}]$. From considerations of symmetry, one obtains a lower confidence bound on β at level $1-\epsilon$ as

$$\omega_n = \min[\sum_{i=1}^{n} T_{i,n}^{\frac{1}{2}} / \sum_{j=1}^{n} (1/T_{j,n}), \quad h'[\ell(\epsilon), T_{1,n}, \ldots, T_{n,n}] \quad (3.2)$$

where $h'[\ell(\epsilon), T_{1,n}, \ldots, T_{n,n}] = [\sqrt{n-1}\, \ell(\epsilon) T_{n,n}^{\frac{1}{2}} - \sum_{i=1}^{n-1} T_{i,n}^{\frac{1}{2}}]/$
$[\sqrt{n-1}\, \ell(\epsilon)/T_{n,n}^{\frac{1}{2}} - \sum_{j=1}^{n-1} (1/T_{j,n})^{\frac{1}{2}}]$.

A test of the hypothesis $H_1: \beta \geq \beta_0$ versus $H_2: \beta < \beta_0$ based on $W_1(\beta_0)$ rejects H_1 at significance level ϵ if

$$\sum_{i=2}^{n} \xi(T_i/\beta_0)/\xi(T_1/\beta_0) > \ell(\epsilon) \quad \text{and} \quad nS(\beta_0) = \sum_{i=1}^{n} \xi(T_i/\beta_0) < 0;$$

a test of $H_2: \beta < \beta_0$ versus $H_1: \beta \geq \beta_0$ based on W_n:

$$W_n(\beta_0) = \frac{1}{\sqrt{n-1}} \sum_{i=1}^{n-1} \xi(T_i/\beta_0)/\xi(T_n/\beta_0)$$

rejects H_2 at significance level ϵ if

$$W_n(\beta_0) \geq \ell(\epsilon) \quad \text{and} \quad nS(\beta_0) > 0.$$

A two-sided test at significance level 2ϵ rejects

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$H_3: \beta_1 \leq \beta \leq \beta_2$ versus $H_4: \beta < \beta_1$ or $\beta > \beta_2$ if

$$nS(\beta_1) > 0 \text{ and } W_n(\beta_2) > \ell(\epsilon) \text{ or } nS(\beta_1) < 0 \text{ and } W_1(\beta_1) > \ell(\epsilon).$$

In like manner one can obtain two-sided confidence bounds at confidence level $1-2\epsilon$ as

$$\omega_n < \beta < \eta_n.$$

Ordinarily, however, since β is the median of the distribution, one would want only a lower bound on β or would want to test only H_1 versus H_2.

Clearly, there are many other invariant functions of the ordered X_i's that one might consider. It is felt, however, that for very small sample sizes and relatively small values of ϵ, it is appropriate to base tests and confidence sets on W_1 and W_n, with constraints either on nS or on $X_{1,n}$ and $X_{n,n}$.

In the following section a Monte Carlo investigation is made of the power of three tests based on W_1 and W_n.

4. Study of Power of Three Proposed Invariant Tests Concerning β

The Monte Carlo study described below was designed to determine which of three proposed procedures would provide most powerful tests, based on very small complete samples, concerning the parameter β. All three procedures depend upon W_1 and W_n, but these statistics are combined differently in each case with constraints on $X_{1,n}$, $X_{n,n}$ or $nS = \sum_{i=1}^{n} X_{i,n}$.

Two of the tests are generalizations of the first two tests in Section 3 applying to a sample of size 2. The third is the test described in Section 3 based on W_n or W_1 and constraints on nS.

The first Monte Carlo simulation procedure described below is designed to determine critical values for testing either $H_1: \beta \geq \beta_0$ versus $H_2: \beta < \beta_0$, H_2 versus H_1, or $H_3: \beta_1 \leq \beta \leq \beta_2$ versus $H_4: \beta < \beta_1$ or $\beta > \beta_2$ at significance level ϵ. In the study we considered samples of size n, n=2,3,4 and 5 and values of ϵ equal to .005, .01, .025, .05, .10. The number N of Monte Carlo samples generated was 10,000.

For each n, N samples $X_1,\ldots,X_n$ of independent standard normal variates were generated using the method of Marsaglia and Bray [4]. For each of the N samples, the values

$$nS = X_1 + \cdots + X_n, \quad X_{1,n} \quad \text{and} \quad X_{n,n}$$

were determined. These values were then used to calculate

$$W_1 = \frac{nS - X_{1,n}}{\sqrt{n-1}\, X_{1,n}} \quad \text{and} \quad W_n = \frac{nS - X_{n,n}}{\sqrt{n-1}\, X_{n,n}}.$$

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The N values of $X_{n,n}$, W_1 and W_n allowed us to determine estimates of

$$D_1(z) = P[X_{n,n} < 0, W_1 > z]$$

$$D_2(z) = P[X_{n,n} < 0, W_n > z]$$

$$D_3(z) = P[nS < 0, W_1 > z]$$

so as to produce a table of $\zeta_k(\epsilon)$ such that $D_k[\zeta_k(\epsilon)] = \epsilon$. The entries of the tables were obtained by letting $X_1(j),\ldots,X_n(j)$ be the jth of the N samples of size n and calculating, e.g. with k = 1,

$$\tilde{D}_1(z) = \frac{1}{N} \sum_{j=1}^{N} \{X_{n,n}(j) < 0, W_1(j) > z\}.$$

Here $\{A\}$ is the indicator of the event A, equal to one if true and zero otherwise. The values $\tilde{D}_k(z_j)$, j=1,...,N were computed from independent samples which form a sequence of independent observations of a decreasing function. Maximum likelihood estimates of D_k were therefore determined by using the method of Brunk [5] for monotone functions applied to the values of $\tilde{D}_k(z_j)$, j=1,...,N. (This technique is also used, e.g., in the construction of maximum likelihood estimators of increasing failure rates, see [6].)

Now, consider the following events:

$A_1 = [X_{n,n} < 0, W_1 > \zeta_1]$, $B_1 = [X_{1,n} > 0, W_n > \zeta_1]$,

$A_2 = [X_{n,n} < 0, W_n > \zeta_2]$, $B_2 = [X_{1,n} > 0, W_1 > \zeta_2]$,

$A_3 = [nS < 0, W_1 > \zeta_3]$, $B_3 = [nS > 0, W_n > \zeta_3]$.

We know from the symmetry of the normal distribution that

$$P(A_k) = P(B_k) = D_k(\zeta_k) = \epsilon, \quad k=1,2,3.$$

Hence, the procedure described above allows one to determine the power of two-sided tests and both one-sided tests on β.

Critical values for testing the various hypotheses of interest were generated under the assumption $\beta_0 = \beta_1 = \beta_2 = \beta$. The following procedure is designed to determine the power of the tests for $\beta_0 = \beta_1 = \beta_2 = \delta^2 \beta$.

First n was fixed and N samples $X_1, \ldots, X_n$ of standard normal variates were generated. The following were then calculated for each ϵ and each j, $j=1,\ldots,N$

$$X'_{1,n}, \quad X'_{n,n}, \quad A', \quad B', \quad W'_1, \quad W'_n,$$

where the primed values denote the fact that T_i/β was replaced by $T_i/(\delta^2\beta)$. Since $X'_i = \xi(T_i/(\delta^2\beta))$, one can determine X'_j as a function of X_i by first solving for T_i/β as a function of X_i and then substituting this function in the expression for X'_i, $i=1,\ldots,n$. Rationaling the denominator, we obtain

$$X'_i = (X_i/2)(\delta + 1/\delta) + (\delta - 1/\delta)\sqrt{1 + X_i^2/4}, \quad i=1,\ldots,n.$$

Then,

$$I_j^{(k)}(\delta) = \{A'_k\} + \{B'_k\}, \quad k=1,2,3,$$

were evaluated (where the prime indicates that primed values of X_i, $i=1,\ldots,n$ are used). Finally, the power functions

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$$P_k(\delta) = \frac{1}{N} \Sigma I_j^{(k)}(\delta), \quad k=1,2,3$$

were computed as functions of δ. The values of δ considered were $\delta = 1/5, 1/4, 1/3, 1/2, 1, 2, 3, 4, 5$.

Monte Carlo simulations of the power curves for two-sided tests at $\epsilon = .01$ for $k=1,2,3$ and all values of $n=2,3,4,5$ are presented in Figures 3, 4 and 5. It can be observed that the tests corresponding to $k = 3$ are more powerful than those corresponding to $k = 1$ for both $n = 4$ and $n = 5$ when testing H_1 versus H_2, H_2 versus H_1 or H_3 versus H_4. It is also apparent that for fixed δ the power tends to increase with n and that the incremental power increase from $k = 1$ to $k = 3$ is larger for $n = 5$ than for $n = 4$.

On the other hand, the sim   ed power curves corresponding to $k = 2$, at $\epsilon = .01$ for va;   values of n, show a markedly different behavior. One can observe for two-sided tests that power first increases as δ deviates from unity in either direction and then decreases as δ approaches either zero or infinity. The tests are therefore not unbiased and consequently not acceptable. Hence no tabulations of the $\zeta_2(\epsilon)$ are presented. Tabulations of $\zeta_1(\epsilon)$, $\zeta_3(\epsilon)$ for $\epsilon = .005, .01, .025, .05, .10$ for $n = 2(1)5$ are given in Table I with the exception that entries are not possible for $\zeta_1(\epsilon)$ whenever $\epsilon < 2^{-n}$. Here $\zeta_k(\epsilon)$ is such that $D_k[\zeta_k(\epsilon)] = \epsilon$ for fixed n and $k = 1,3$.

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Table I

Critical values of $\zeta_k(\epsilon)$ computed for $N = 10,000$

n = 2

ϵ	.005	.01	.025	.05	.10
$\zeta_1(\epsilon)$	.97	.94	.84	.74	.51
$\zeta_3(\epsilon)$	.97	.94	.85	.74	.51

n = 3

ϵ	.005	.01	.025	.05	.10
$\zeta_1(\epsilon)$	1.12	1.03	.86	.68	.33
$\zeta_3(\epsilon)$	1.13	1.04	.85	.67	.44

n = 4

ϵ	.005	.01	.025	.05	.10
$\zeta_1(\epsilon)$	1.16	1.01	.78	.47	x
$\zeta_3(\epsilon)$	1.15	1.02	.81	.65	.43

n = 5

ϵ	.005	.01	.025	.05	.10
$\zeta_1(\epsilon)$	1.11	.93	.56	x	x
$\zeta_3(\epsilon)$	1.11	1.00	.81	.64	.41

An illustration of the use of Table I is now given. Let $t_1, \ldots, t_5$ be the ordered failure times of a device in life testing. Under the assumption that the unordered re times would be a random sample from the two-parameter distribution defined by Equation (0.1), a $100(1-\epsilon)\%$ lower confidence bound on β is given by Equation (3.1), namely

$$\min \left[\sum_{i=1}^{5} t_i^{\frac{1}{2}} \Big/ \sum_{i=1}^{5} t_i^{-\frac{1}{2}}, \quad \frac{2\zeta_3(\epsilon)\sqrt{t_n} - \sum_{1}^{4} t_i^{\frac{1}{2}}}{2\zeta_3(\epsilon)t_n^{-\frac{1}{2}} - \sum_{1}^{4} t_i^{-\frac{1}{2}}} \right]$$

Suppose the life times, in hours, were

$$t_1 = 48,310$$

$$t_2 = 55,154 \qquad t_3 = 61,273$$

$$t_4 = 58,110 \qquad t_5 = 67,769$$

and we set $\epsilon = .1$. We compute a lower confidence bound for β at level .90 to be equal to 45,375.
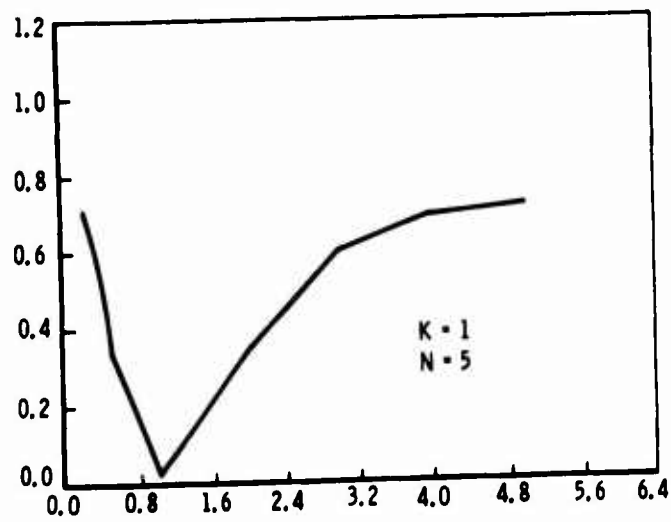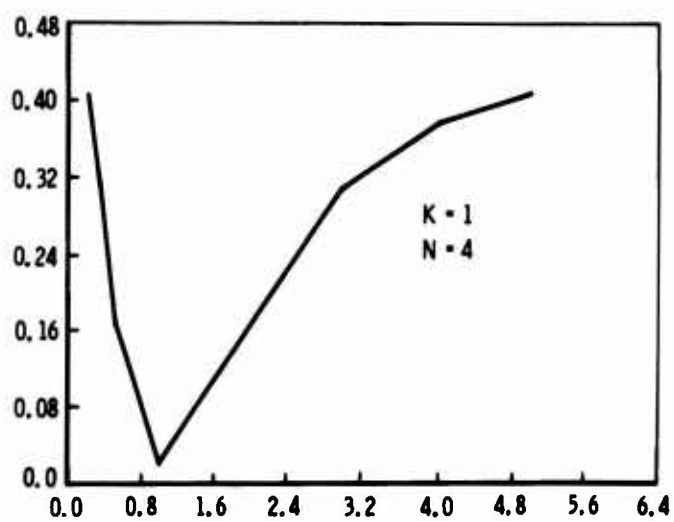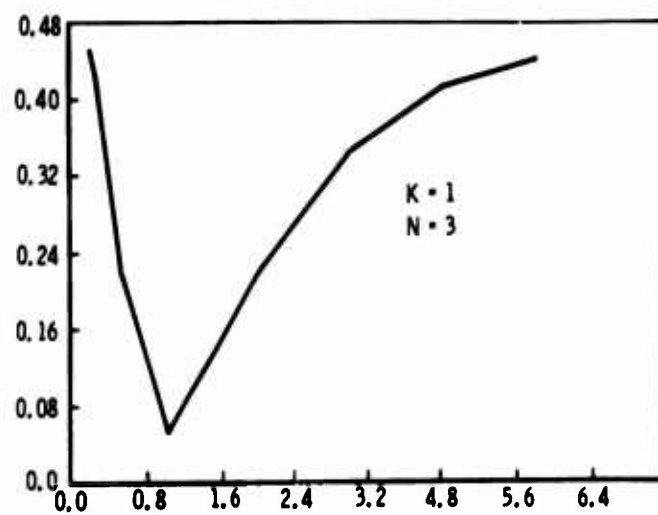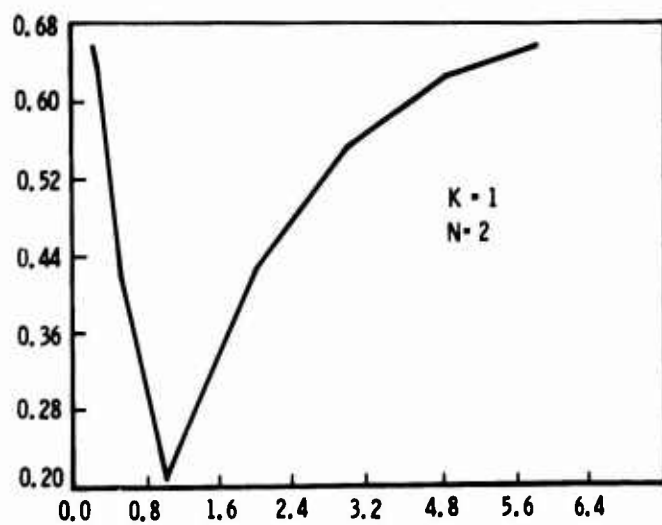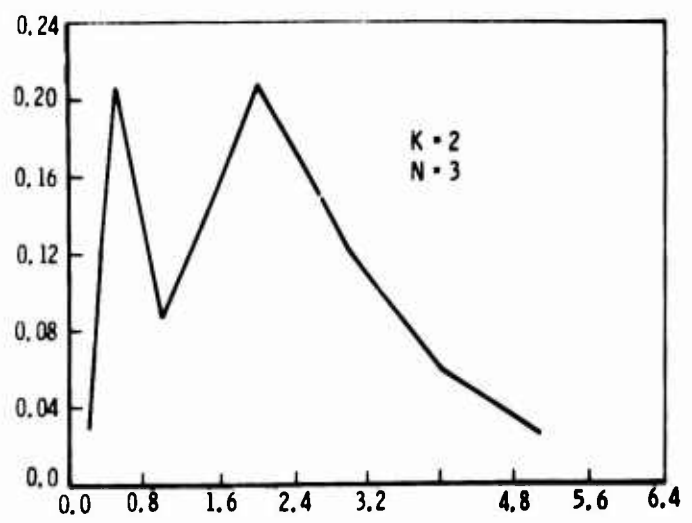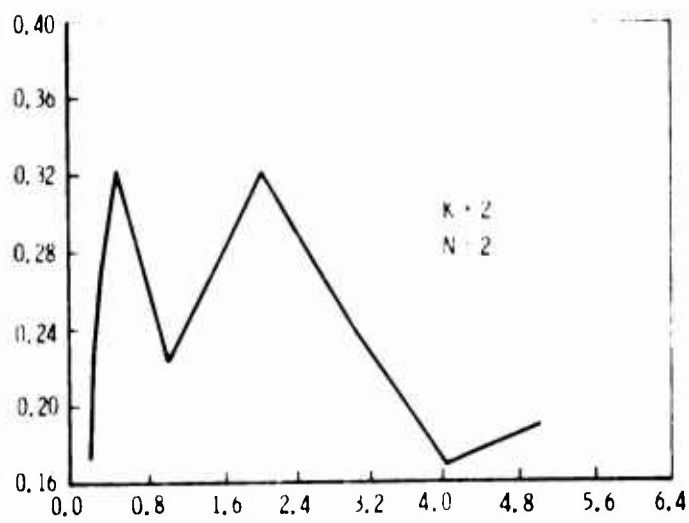
Figure 3. Graphs of power as a function of δ.
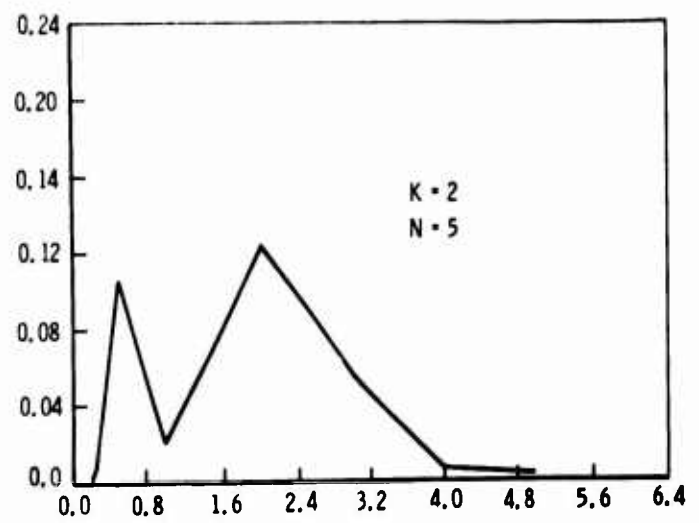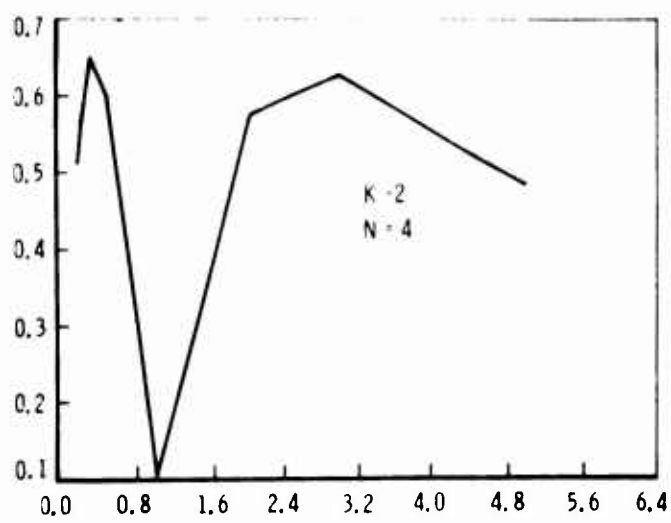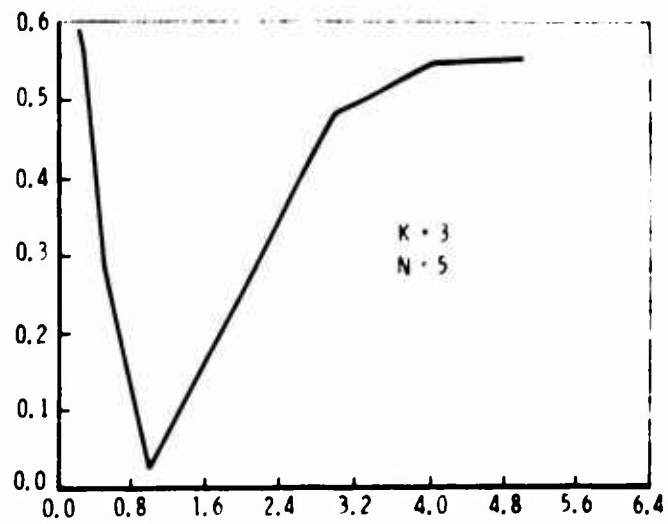
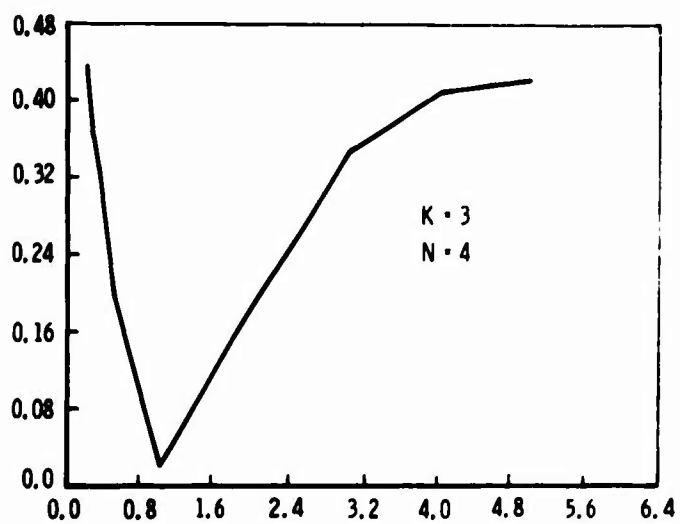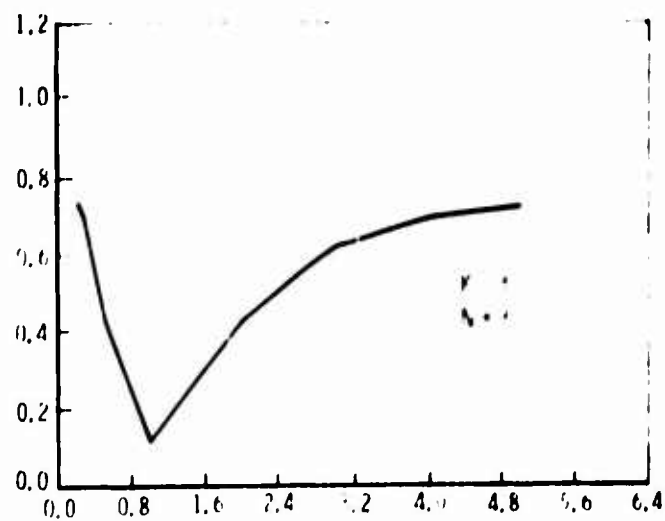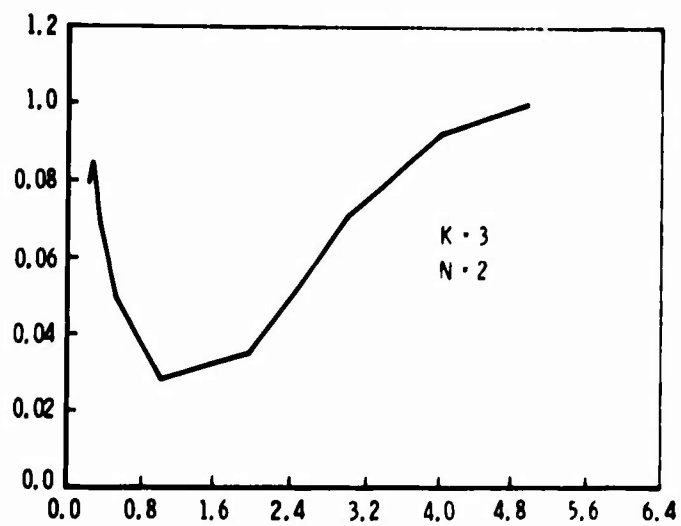Figure 4. Graphs of power as a function of δ.

Figure 5. Graphs of power as a function of .

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     Distributions", *Journ. Appl. Probab.*, <u>6</u>, 319-327 (1969).

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